The Report committee for Anna Marie Bouboulis

Certifies that this is the approved version of the following report:

Poincaré Disc Models in Hyperbolic Geometry

APPROVED BY

SUPERVISING COMMITTEE:

Supervisor: ______________________________
John Luecke

____
Mark Daniels

Poincaré Disc Models in Hyperbolic Geometry

by

Anna Marie Bouboulis, BA

Report

Presented to the Faculty of the Graduate School

of the University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Master of Arts

The University of Texas at Austin

August 2013

Abstract

Poincaré Disc Models in Hyperbolic Geometry

Anna Marie Bouboulis, MA

The University of Texas at Austin, 2013

SUPERVISOR: John Luecke

This report discusses two examples of the use of Poincaré disc models and their different relationships to Euclidean geometry. The topics include light reflection in hyperbolic geometry and the Hyperbolic Pythagorean Theorem; all in relation to the Poincaré unit disc model and Poincaré upper half plane model.

Table of Contents

List of Figures ……………………………………………………………………………………………………vi

Chapter One: Introduction………………………………………………………………………………….1

Chapter Two: Hyperbolic Light Ray Reflection……………………………………………………….3

Chapter Three: Hyperbolic Pythagorean Theorem………………………………………………….10

Chapter Four: Conclusion……………………………………………………………………………….15

References…………………………………………………………………………………………………….17

List of Figures

Figure 1: Reflection of light rays off a curve ……………………………………………………………3

Figure 2: Slope of the line tangent to the curve..4

Figure 3: Model for equation of the semicircle…………………………………………………………5

Figure 4: Equation of semicircle reflecting off a curve…………………………………………………7

Figure 5: Hyperbolic triangle before and after a Möbius transformation………………………12

Chapter One: Introduction

Geometry can be categorized into two main subcategories, Euclidean and non-Euclidean geometry. The difference between these two categories stems from the Parallel Postulate defined by Euclid in his book *Elements.*[**1**] By using different versions of the parallel postulate, the different geometries are formed. As specific subcategories of geometry might be more useful with different topics, being able to have analogous properties between the different geometries also becomes helpful.

One way to relate the different geometries is to use a model that uses properties of a non-Euclidean geometry in a Euclidean analytical manner. The Poincaré upper half-plane model and the Poincaré unit disc model are two models that can be used to relate Euclidean analysis with hyperbolic geometry. Poincaré's models will be used to identify a hyperbolic analog to reflective properties of curves and to the side length relationship of right triangles that one sees in Euclidean geometry.

The first example illustrates a similarity between the behavior of light rays in Euclidean geometry and light rays in hyperbolic geometry. When light rays are reflected off a parabolic curve in Euclidean geometry, the reflected rays are focused to a single point. An analog in hyperbolic geometry will be found. The Poincaré upper half plane model is used to identify the aspects of similarity because this model preserves angles between Euclidean and hyperbolic geometry. Euclidean light rays are vertical lines and arcs of semicircles when using the Poincaré upper half plane model. The equation of the semicircles which represent the reflected light rays will be written with respect to the angles and slopes created by the curve and light rays. By solving the differential equation created, it is found that hyperbolas and ellipses are the reflective curves in hyperbolic geometry that have the characteristic of reflecting vertical light rays to a single point.

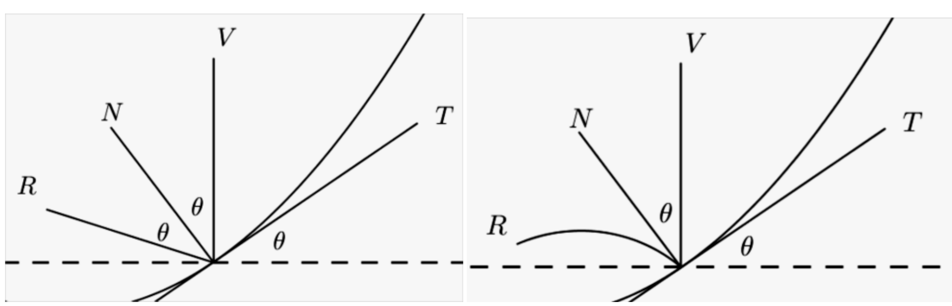
The Pythagorean Theorem, in Euclidean geometry, states that the two perpendicular sides of a right triangle can be squared and added to equal the square of the side opposite the right angle. To find the equivalent statement that relates the sides of a right triangle in hyperbolic geometry the Poincaré unit disc model is used. In this model the sides of the triangle are in fact sub-arcs of semicircles that are perpendicular to the unit disc. The use of Möbius transformations formats the hyperbolic triangle so that the sides and shape can be written as vectors and thus produces a Hyperbolic Pythagorean Theorem that is similar to the Pythagorean Theorem in Euclidean geometry.

These examples do not exhaust the possible uses of Poincaré's different models, nor are these the only models that relate Euclidean geometry to hyperbolic geometry. They are two different and interesting relationships between the two geometries and each model has an isomorphism that can transfer between the two models. In the next chapters, these models are applied to prove the above analogs of theorems from Euclidean geometry.

Chapter Two: Hyperbolic Light Ray Reflection

The Poincaré upper half-plane model will be used to show that vertical light rays reflected off a specific type of curve reflect to a single point. Light rays are represented by vertical lines, perpendicular to the x-axis and parallel to the y-axis. All other hyperbolic lines, or geodesics, are semicircles that are centered on the x-axis. The Poincaré upper half-plane model is conformal to Euclidean geometry and as the law of reflection uses the angles between lines, it can also be applied in the Poincaré half-plane model. "The law of reflection [states that] when light rays encounter a reflection curve, the angle of incidence equals the angle of reflection." [**2**, p. 377] (Figure 1)

The equation of the reflected light rays will be written in a form that allows a family of curves for the reflective curve to be identified by solving a differential equation.



a) Euclidean Geometry b) Hyperbolic Geometry

Figure 1: Reflection of light rays off a curve

First, the equations for the light rays must be identified. The following are all related by the angle of incidence. The vertical light rays (V) are perpendicular to the x-axis. The reflective curve is $y = f(x)$. The line tangent to the reflective curve (T), the normal ray (N), which is perpendicular to the tangent line, and the reflective ray (R) can be drawn as seen in Figure 1 for Euclidean and hyperbolic geometry.

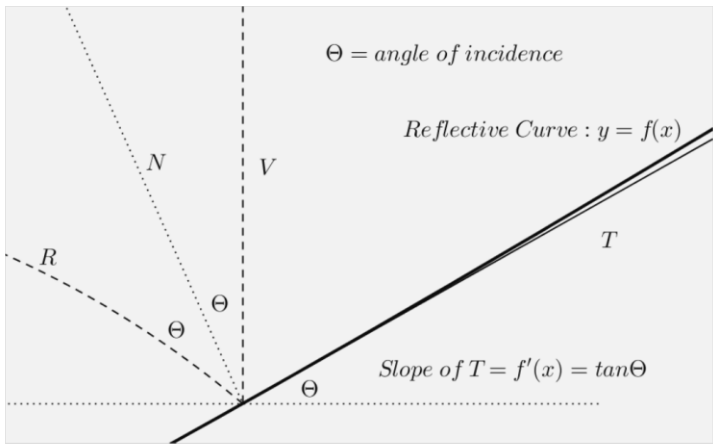


Figure 2: Slope of the line tangent to the curve

Focusing on the hyperbolic geometry relationships, see figure 2, the slope of T can be written as $f'(x) = tan\ \theta$ and the slope of the reflective ray (R) as

$$m_R = \tan\left(\frac{\pi}{2} + 2\theta\right) = -\cot(2\theta) = \frac{\tan^2\theta - 1}{2tan\theta} = \frac{\left(f'(x)\right)^2 - 1}{2f'(x)}. \qquad (1)$$

Equation (1) will be used to write the equation for the path of the reflected ray, which is a semi-circle, in terms of the slope of the reflected path.

To determine the equation of the semicircles in the Poincaré upper half-plane model the following must be true for any semicircle; the center is on the x-axis $(x_c, 0)$, there is a point $p(x_0, y_0)$ on the circle with a tangent line at p that has a slope of m. (Figure 3)

The basic form of the equation of a semicircle,

$$R^2 = (x_0 - x_c)^2 + y_0^2 \qquad (2)$$

can be written in terms of slope m by utilizing that the line perpendicular to the tangent line, the normal line (N), intersects the x-axis at the center of the semicircle such that

$$x_c = my_0 + x_0 \qquad (3)$$

and

$$(x_c - x_0)^2 = (my_0)^2. \quad (4)$$

By substituting (4) and manipulating the equation the equation of the semicircle (2) can be written as:

$$R^2 = (m^2 + 1)y_0^2. \qquad (5)$$

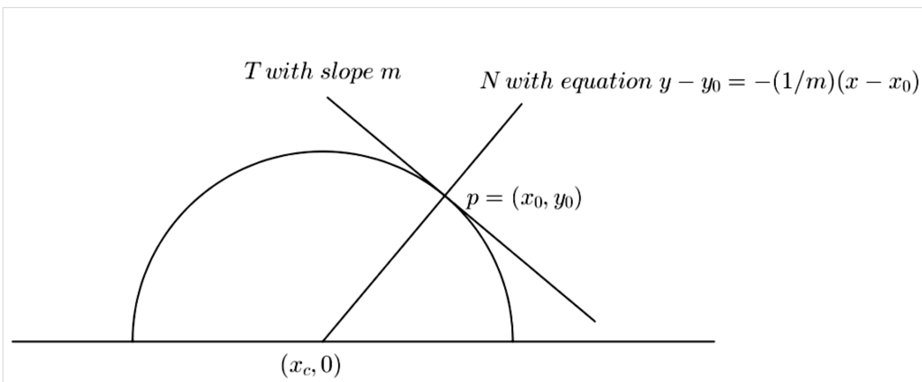


Figure 3: Model for equation of the semicircle

As the R^2 can be written as the distance between the center of the circle and point p the following is true:

$$(x_0 - x_c)^2 + y_0^2 = (m^2 + 1)y_0^2. \quad (6)$$

Equation (2) is satisfied by any point (x, y) on the same semi-circle. Using the expression of R in terms of (x_0, y_0) we get

$$(x - x_c)^2 + y^2 = (m^2 + 1)y_0^2. \qquad (7)$$

This uses (1) which already means that $f(x)$ is a reflected curve. Thus if $y = f(x)$ has the property that (x_0, y_0) it reflects a vertical light ray to a point $(0, y_I)$ on the y-axis, then it must satisfy (9).

$$y_I^2 = y_0^2 - 2mx_0y_0 - x_0^2 \qquad (8)$$

By writing (x_0, y_0) as $(x_0, f(x_0))$ and using (1) as a substitution of m , (8) can be written as

$$y_I^2 = f(x_0)^2 + \frac{\left(\left(f'(x_0)\right)^2 - 1\right)}{f'(x_0)}(x_0)f(x_0) - x_0^2. \qquad (9)$$

Thus, now we have an equation for the semicircles that are centered on the x-axis and have the y-intercept as a common point. This equation will thought of as giving a differential equation that determines the reflective curve, $y = f(x)$.

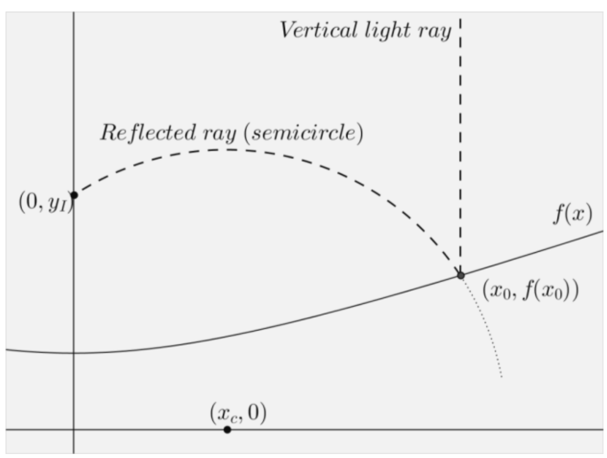


Figure 4: Equation of semicircle reflecting off a curve

Let $f(x)$ have the property of reflecting vertical light rays to a single point and let this focal point lie on the y-axis. (Figure 4) With this condition it can be assumed that

y_I is independent of x_0. Let c be the focal point and use it to replace the notation for the y-intercept in (9). Thus c and $f(x)$ satisfy the following equation. [**2**, 379]

$$c^2 = f(x)^2 + \left(\frac{1}{f'(x)}\right)(f(x)'^2 - 1)\big(xf(x)\big) - x^2. \quad (10)$$

A change in notation is used to better format (10) for solving, $f(x)$ is changed to $y(x)$ and manipulation after produces the form below which is the differential equation.

$$\frac{c^2}{xy} + \frac{x}{y} - \frac{y}{x} = \frac{1}{y'} - y' \qquad (11)$$

If in (11) y' is replaced with $\frac{-1}{y'}$ the same equation is obtained, which implies that orthogonal curves will also have the same reflective property. Also if (11) is multiplied by y' it can be shown to form a quadratic expression in y' and when solving the quadratic expression two equations are obtained, $y' = f_1(x, y)$ and $y' = f_2(x, y)$. This can be used to justify that the solution to the differential equation is a set of two curves that are orthogonal, $y' = f(x, y)$and $y' = \frac{-1}{f(x,y)}$. Conics are the plausible solution curve as they produce orthogonal curves that have a common focal point under Euclidean geometry. [**2**, p.379]

To show that conics are in fact the family of curves needed, the basic conic equation is used to obtain a differential equation with y' as the solution to equation (11). Let $\lambda(x, y) = k$, defining y as a function of x and with $b^2 > a^2 > 0$ fixed.

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \qquad (12)$$

The equation above can be implicitly differentiated with respect to x and y respectively

$$\frac{2x}{a^2+\lambda} = \lambda_x \left(\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2}\right) \qquad (13)$$

and

$$\frac{2y}{b^2+\lambda} = \lambda_y \left(\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2}\right), \qquad (14)$$

thus obtaining the level of curves of λ, which form families of ellipses and hyperbolas.

Thus $grad\,\lambda = (\lambda_x, \lambda_y)$ is proportional to the vector $\left(\frac{x}{a^2+\lambda}, \frac{y}{b^2+\lambda}\right)$. Using properties of gradients and the vector tangent to the curve $\lambda = k$, $\left(-\frac{y}{b^2+\lambda}, \frac{x}{a^2+\lambda}\right)$, it can be determined that the ratio $-\left(\frac{x}{y}\right)\left(\frac{b^2+\lambda}{a^2+\lambda}\right)$ is equal to $y'(x)$. This gives

$$\frac{1}{y'} - y' = \frac{-\left(\frac{y}{x}\right)(a^2+\lambda)}{(b^2+\lambda)} + \frac{\left(\frac{x}{y}\right)(b^2+\lambda)}{(a^2+\lambda)} \qquad (15)$$

Equation (15) can be manipulated to give equation (16), which is the same as equation (11) when $c^2 = b^2 - a^2$.

$$\frac{1}{y'} - y' = \frac{-\left(\frac{y}{x}\right)(a^2-b^2+b^2+\lambda)}{(b^2+\lambda)} + \frac{\left(\frac{x}{y}\right)(b^2-a^2+a^2+\lambda)}{(a^2+\lambda)}$$

$$\frac{1}{y'} - y' = \frac{-\left(\frac{y}{x}\right)(a^2-b^2)}{(b^2+\lambda)} - \frac{\left(\frac{y}{x}\right)(b^2+\lambda)}{(b^2+\lambda)} + \frac{\left(\frac{x}{y}\right)(b^2-a^2)}{(a^2+\lambda)} + \frac{\left(\frac{x}{y}\right)(a^2+\lambda)}{(a^2+\lambda)}$$

$$\frac{1}{y'} - y' = \frac{\left(\frac{y}{x}\right)(b^2-a^2)}{(b^2+\lambda)} - \left(\frac{y}{x}\right) + \frac{\left(\frac{x}{y}\right)(b^2-a^2)}{(a^2+\lambda)} + \left(\frac{x}{y}\right)$$

$$\frac{1}{y'} - y' = \frac{y^2(b^2-a^2)}{xy(b^2+\lambda)} - \left(\frac{y}{x}\right) + \frac{x^2(b^2-a^2)}{xy(a^2+\lambda)} + \left(\frac{x}{y}\right)$$

$$\frac{1}{y'} - y' = \left(\frac{b^2-a^2}{xy}\right)\left(\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)}\right) - \left(\frac{y}{x}\right) + \left(\frac{x}{y}\right)$$

$$\frac{1}{y'} - y' = \left(\frac{b^2-a^2}{xy}\right)(1) - \left(\frac{y}{x}\right) + \left(\frac{x}{y}\right)$$

$$\frac{1}{y'} - y' = -\frac{y}{x} + \frac{x}{y} + \frac{b^2-a^2}{xy} \qquad (16)$$

Thus it is shown that not only can light be reflected off a curve to a single foci in hyperbolic geometry, but the curve needed for this specific type of reflection is from the family of hyperbolas and ellipses.

Chapter Three: Hyperbolic Pythagorean Theorem

A right triangle, in Euclidean geometry, has a property that the sum of each of the two perpendicular side lengths squared is equal to the third side squared. In hyperbolic geometry the same property is not true, yet an analogous property can be found using the *Poincaré unit disc model*.

Poincaré's unit disc model takes a set of all complex numbers with the norm length less than 1,

$$\mathcal{D}^2 = \{z \in \mathbb{C} \mid |z| < 1\}.$$

In this model straight lines are represented as all segments of diameters or semicircles that are orthogonal to the boundary circle. [**3,** pg. 760] The angles between curves in hyperbolic space are the same as the Euclidean angles. Möbius transformations preserve hyperbolic length and angle measure. Restricting to those Möbius transformations that take the unit disc to the unit disc. These Möbius transformation defines Möbius addition in the disc model.

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z) \qquad (17)$$

This allows for the Möbius transformations to be described as Möbius left translation followed by a rotation, [**3**, p. 759]

$$z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}. \qquad (18)$$

In (17) and (18) $\theta \in \mathbb{R}$ is a real number, $z_0 \in \mathcal{D}$, and $\bar{z}_0$ is the complex conjugate of z_0. [**3**, p. 759] The Poincaré hyperbolic distance function in $\mathcal{D}$ is given by

$$d(a,b) = \left|\frac{a-b}{1-\bar{a}b}\right| = |a \ominus b|, \qquad (19)$$

where $a \ominus b = a \oplus (-b)$ for $a, b \in \mathcal{D}$.

Since hyperbolic length and angle measures are preserved with the Möbius transformations, moving (matching) the values of the angles of a triangle and lengths of the sides of a triangle so they line up on the real and imaginary axes of the unit disc provides a way to simplify the values to get the analogous form of the Pythagorean theorem.

Theorem. (The Hyperbolic Pythagorean Theorem) *Let Δabc be a hyperbolic triangle in the Poincaré disc, whose vertices are the points a, b, and c of the disc and whose sides (directed counterclockwise) are $A = -b \oplus c,\ B = -c \oplus a,\ and\ C = -a \oplus b.$ If the two sides A and B are orthogonal, then $|A|^2 \oplus |B|^2 = |C|^2$.* [**3,** p. 760]

In the proof for this Hyperbolic Pythagorean Theorem it is necessary to understand that the sides A, B, and C are geodesic segments that join the vertices and the measure of an angle in a hyperbolic triangle is the "Euclidean measure of the angle formed by Euclidean tangent rays." [**3,** p.760] Using a Möbius transformation the hyperbolic right triangle Δabc can be moved, such that the two orthogonal sides lie on the real and imaginary axes of D in the Poincaré disc model, forming $\Delta a'b'c'$.[Figure 3]

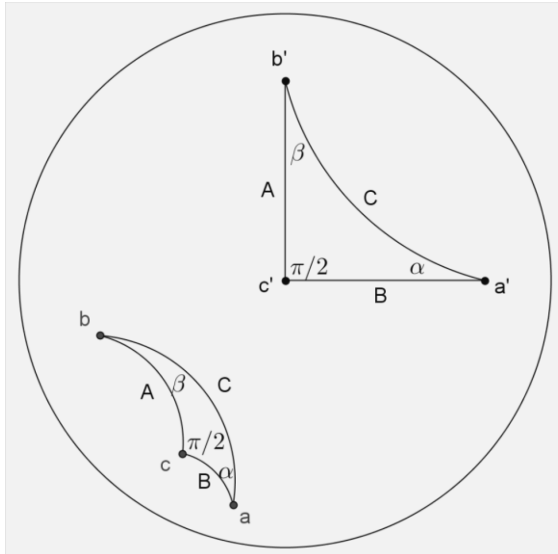


Figure 5: Hyperbolic triangle before and after a Möbius transformation

As illustrated in figure 5 the new vertices and lengths of the hyperbolic segments after the Möbius transformation are

$$a' = x,\ \ b' = iy,\ \ c' = 0. \qquad (20)$$

The side lengths of the triangle are $|A|, |B|, and\ \ |C|$, using the definition of the norm of a complex number $|z\,| = \sqrt{\,(e)^2 + (f)^2}$ where $z = e + fi$. Using the values for $a', b', and\ c',$ the definition of distance (16) in the Poincaré disc model and the norm, the side lengths squared can be written as:

$$|A|^2 = |b' \ominus c'|^2$$

$$\left|\frac{iy - 0}{1 - (-iy)(0)}\right|^2 = \left|\frac{iy}{1}\right|^2 = \left(\sqrt{y^2}\right)^2 = y^2$$

$$|A|^2 = y^2. \qquad (21)$$

Similarly,

$$|B|^2 = |a' \ominus c'|^2 = \left(\sqrt{x^2}\right)^2$$

$$|B|^2 = x^2 \qquad (22)$$

and

$$|C|^2 = |a' \ominus b'|^2 = |x \ominus iy|^2 = \left|\frac{x-iy}{1-iyx}\right|^2. \quad (23)$$

The manipulation of the fraction in (23) is key to showing it can be written in the needed form. Below is the process to show $|C|^2 = |A|^2 \oplus |B|^2$:

$$\left(\frac{x-iy}{1-iyx}\right) = \left(\frac{x-iy}{1-iyx}\right)\left(\frac{1+ixy}{1+ixy}\right) = \left(\frac{x+xy^2-iy+ix^2y}{1+x^2y^2}\right) \quad (24)$$

Equation (24) can be written as:

$$\left(\frac{x+xy^2}{1+x^2y^2}\right) + \left(\frac{x^2y-y}{1+x^2y^2}\right)i \qquad (25)$$

Using (25) with the norm,

$$\left|\left(\frac{x+xy^2}{1+x^2y^2}\right) + \left(\frac{x^2y-y}{1+x^2y^2}\right)i\right|^2 = \left(\sqrt{\left(\frac{x+xy^2}{1+x^2y^2}\right)^2 + \left(\frac{x^2y-y}{1+x^2y^2}\right)^2}\right)^2$$

Continue simplifying to finally get,

$$\frac{\left(x+xy^2\right)^2}{(1+x^2y^2)^2} + \frac{(x^2y-y)^2}{(1+x^2y^2)^2} = \frac{\left(y^2+x^2\right)(1+y^2x^2)}{(1+y^2x^2)(1+y^2x^2)} = \frac{y^2+x^2}{1+y^2x^2} = y^2 \oplus x^2$$

Lastly, substituting in the notation for the corresponding values yields,

$$|C|^2 = |A|^2 \oplus |B|^2, \quad (25)$$

which is the Hyperbolic Pythagorean Theorem for a hyperbolic right triangle in the Poincaré unit disc model. This shows that by way of using Möbius transformations and vector properties produce an analog situation in hyperbolic geometry for the Pythagorean Theorem.

Chapter Four: Conclusion

Poincaré's upper half plane model and unit disc model discussed previously presents useful ways of relating topics known in Euclidean geometry to hyperbolic geometry. Euclidean analytical geometry and trigonometry can be used to show that the hyperbolic axioms hold. [**4**] Illustrated in Chapter Two an analytical approach to writing the equations of the vertical light rays forms a differential equation that can be solved to find the analog curve. The substitutions involving different forms of slope and the equations of circles is a concept that is taught in high school. The manipulation of these equations and comparison of the slopes of the different equations provide a way to approach this advanced topic with younger students possessing a less varied mathematics background.

As high school curriculum traditionally requires the Pythagorean Theorem to be taught, having the students relate two triangles that look very different to similar equations can help with the comprehension of properties inherent to hyperbolic geometry. The Poincaré unit disc model is not initially easily understood. Treating the unit disc as a vector space provides an avenue for an analytical approach. While the Möbius transformations are a difficult topic, using a vector approach is a possible way to help further the understanding of hyperbolic space. Using the Poincaré hyperbolic distance function and the formula to find the length of a vector, students should be able to see the relationship between Euclidean triangles with those in hyperbolic space. The more these two geometries can be shown to relate the more student understanding of high school geometry can be enhanced.

References

1. Greenberg, Marvin Jay. "Euclidean and Non-Euclidean Geometries: Development and History". *New York: W.H Freeman and Company.* (1993), 9-443.

2. Perline, Ron. "Non-Euclidean Flashlights". *The American Mathematical Monthly,* 103 (May, 1996): 337-385.

3. Ungar, Abraham A. "The Hyperbolic Pythagorean Theorem in the Poincaré Disc Model of Hyperbolic Geometry". *The American Mathematical Monthly,* 106 (Oct., 1999): 759-763.

4. Millman, Richard. "The Upper Half Plane Model for Hyperbolic Geometry". *The American Mathematical Monthly.* 87.1 (Jan., 1980): 48-53.

5. Birman, G., & Ungar, A. "The Hyperbolic Derivative in the Poincaré Ball Model of Hyperbolic Geometry". *Journal of Mathematical Analysis and Applications.* 253. (2001), 321-333.
doi:10.1006/jmaa.200.7280